OCTOGON MATHEMATICAL MAGAZINE
Vol. 31, No.1, April 2023, pp 100-105
Print: ISSN 1222-5657, Online: ISSN 2248-1893
http://www.uni-miskolc.hu/~matsefi/Octogon/

One ring of inequalities

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At the beginning we consider independently several inequalities which appeared in different time in different published materials.
Here is the list of these inequalities:

1. Sharp Quadratic Mix Inequality

$$a^2+b^2+c^2-ab-bc-ca \geq \sqrt[3]{abc}\left(a+b+c-3\sqrt[3]{abc}\right), a,b,c>0 \quad (SQM)$$

2. Turkevich's Inequality [3]

$$x^4+y^4+z^4+t^4+2xyzt \geq\geq$$

$$x^2y^2+y^2z^2+z^2t^2+t^2x^2+x^2z^2+y^2t^2, x,y,z,t>0. \quad (T)$$

3. Dospinescu-Lascu-Tetiva inequality [1]

$$a^2+b^2+c^2+2abc \geq (1+a)(1+b)(1+c), a,b,c>0. \quad (DLT)$$

4. Darij Grinberg Inequality [2]. pr.5, p.5

$$1+2abc+a^2+b^2+c^2 \geq 2ab+2bc+2ca, a,b,c \geq 0. \quad (DG)$$

Proofs.
1. Proof of (SQM). Due to homogeneity assume that $a+b+c=1$ then, denoting $p:=ab+bc+ca, q:=abc,$ obtain (SQM)

$$\Longleftrightarrow 1-3p \geq \sqrt[3]{q}\left(1-3\sqrt[3]{q}\right).$$

[8]Received: 14.03.2023
2010 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Inequalities.

Since

$$p \le \frac{1+9q}{4} \iff 1 \ge 4p - 9q$$

(Schur's Inequality $\sum_{cyc} a(a-b)(a-c) \ge 0$ in p-q notation) and

$q = abc \le \left(\frac{a+b+c}{3}\right)^3 = \frac{1}{27}$ suffices to prove

$$1 - \frac{3(1+9q)}{4} \ge \sqrt[3]{q}\,(1 - 3\sqrt[3]{q}) \iff 1 - 27q \ge 4\sqrt[3]{q}\,(1 - 3\sqrt[3]{q})$$

for any $q \in \left[0, \frac{1}{27}\right]$.

We have

$$1 - 27q - 4\sqrt[3]{q} + 12\sqrt[3]{q^2} = (1 - 3\sqrt[3]{q})\left(1 + \sqrt[3]{q} + \sqrt[3]{q^2}\right) - 4\sqrt[3]{q}\,(1 - 3\sqrt[3]{q}) =$$

$$(1 - 3\sqrt[3]{q})\left(\left(1 - 3\sqrt[3]{q} + \sqrt[3]{q^2}\right)\right) = (1 - 3\sqrt[3]{q})^2 + \sqrt[3]{q^2}\,(1 - 3\sqrt[3]{q}) \ge 0.$$

Remark. *By setting* $x = \sqrt[3]{\frac{a^2}{bc}}, x = \sqrt[3]{\frac{b^2}{ca}}, x = \sqrt[3]{\frac{c^2}{ab}}$ *in (SQM), we obtain*

$$\sum_{cyc} \sqrt[3]{\frac{a^4}{b^2c^2}} + 3 \ge \sum_{cyc} \sqrt[3]{\frac{bc}{a^2}} + \sum_{cyc} \sqrt[3]{\frac{a^2}{bc}} \iff$$

$$x^2 + y^2 + z^2 + 3 \ge x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

which holds for any $x, y, z > 0$ *such that* $xyz = 1$.

3. Proof of (DLT). Since

$$a^2 + b^2 + c^2 + 2abc + 3 \ge (1+a)(1+b)(1+c) \iff$$

$$a^2 + b^2 + c^2 + 2abc + 3 \ge 1 + a + b + c + ab + bc + ca + abc \iff$$

$$a^2 + b^2 + c^2 - 2a - 2b - 2c + 3 \ge 1 - a - b - c + ab + bc + ca - abc \iff$$

$$(1-a)^2 + (1-b)^2 + (1-c)^2 \ge (1-a)(1-b)(1-c).$$

then denoting $x := 1 - a, y := 1 - b, z := 1 - c$ we obtain for original inequality new equivalent form

$$x^2 + y^2 + z^2 \geq xyz \tag{B}$$

where $x, y, z < 1$.
Since for $xyz \leq 0$ this inequality obviously holds, then enough to prove it for $xyz > 0$.In that case at least one of the variables is positive. Let it be z. Then $z \in (0, 1)$, $xy > 0$ and

$$x^2 + y^2 + z^2 \geq 2xy + z^2 > 2xyz + z^2 > xyz.$$

Equality holds iff $x = y = z = 0$.

4. Proof of (DG). Since

$$1 + 2abc + a^2 + b^2 + c^2 \geq 2ab + 2bc + 2ca \iff$$

$$1 + 2abc + (a + b + c)^2 \geq 4(ab + bc + ca) \iff$$

$$(a + b + c)(1 + 2abc) + (a + b + c)^3 \geq 4(ab + bc + ca)(a + b + c)$$

and by Schur's Inequality

$$(a + b + c)^3 + 9abc \geq 4(ab + bc + ca)(a + b + c)$$

then suffices to prove inequality

$$(a + b + c)(1 + 2abc) \geq 9abc.$$

This inequality immediately follows from $a + b + c \geq 3\sqrt[3]{abc}$ and

$$\sqrt[3]{abc}(1 + 2abc) \geq 3abc \iff 1 + 2abc - 3\sqrt[3]{a^2b^2c^2} \geq 0 \iff$$

$$\left(\sqrt[3]{abc} - 1\right)^2 \left(2\sqrt[3]{abc} + 1\right) \geq 0.$$

Consider now another proof of inequality (DG).

$$1 + 2abc + a^2 + b^2 + c^2 \geq 2ab + 2bc + 2ca \iff$$

$$3+a^2+b^2+c^2-2(a+b+c) \geq 2-2(a+b+c)+2ab+2bc+2ca-2abc \iff$$

$$(1-a)^2+(1-b)^2+(1-c)^2 \geq$$

$$\geq 2(1-a)(1-b)(1-c) \iff \quad x^2+y^2+z^2 \geq 2xyz,$$

where $x = 1-a < 1, y = 1-b < 1, z = 1-c < 1$.
Consider two cases:
1. $xyz \leq 0$ then inequality obvious;
2. Let $xyz > 0$, then if $x, y, z > 0$ we have $x, y, z \in (0, 1]$ and, therefore,

$$x^2+y^2+z^2-2xyz = (x-yz)^2+y^2+z^2-y^2z^2 = (x-yz)^2+y^2\left(1-z^2\right)+z^2 \geq 0;$$

if $x > 0$ and $yz > 0$ then $x \in (0, 1]$ and

$$x^2+y^2+z^2-2xyz \geq x^2+2yz(1-x) \geq 0.$$

Thus (DG) equivalent to inequality

$$x^2+y^2+z^2 \geq 2xyz, \tag{B1}$$

where $x, y, z < 1$.

Remark. *Note that inequality (B1) stronger then (B) .*

Indeed, if $xyz \leq 0$ then (B) and (B1) holds simultaneously; if $xyz > 0$ then

$$x^2+y^2+z^2 \geq 2xyz \geq xyz.$$

Replacing (a, b, c) in (DG) with $\left(\dfrac{a}{\sqrt[3]{abc}}, \dfrac{b}{\sqrt[3]{abc}}, \dfrac{b}{\sqrt[3]{abc}}\right)$ or (x, y, z) in (B) with $\left(1-\dfrac{a}{\sqrt[3]{abc}}, 1-\dfrac{b}{\sqrt[3]{abc}}, 1-\dfrac{b}{\sqrt[3]{abc}}\right)$ we obtain another Sharp Quadratic Mix Inequality (SQM1)

$$a^2+b^2+c^2-ab-bc-ca \geq ab+bc+ca-3\sqrt[3]{a^2b^2c^2}.$$

Theorem. $(B) \implies (SQM) \implies (T) \implies (DLT) \implies (B)$.

Proof. $(B) \implies (SQM):$

Let $x = 1 - u,\ y = 1 - v,\ z = 1 - w$ then $u, v, w > 0$ and

$$(1-u)^2 + (1-v)^2 + (1-w)^2 \geq (1-u)(1-v)(1-w) \iff$$

$$u^2 + v^2 + w^2 + 2 + uvw \geq u + v + w + uv + vw + wu.$$

In particularly for $u := \dfrac{a}{\sqrt[3]{abc}}, v := \dfrac{b}{\sqrt[3]{abc}}, w := \dfrac{c}{\sqrt[3]{abc}}$ where $a, b, c > 0, p$ since $uvw = 1,$ we obtain

$$\frac{a^2}{\left(\sqrt[3]{abc}\right)^2} + \frac{b^2}{\left(\sqrt[3]{abc}\right)^2} + \frac{c^2}{\left(\sqrt[3]{abc}\right)^2} + 3 \geq \frac{a}{\sqrt[3]{abc}} + \frac{b}{\sqrt[3]{abc}} + \frac{c}{\sqrt[3]{abc}} +$$

$$\frac{ab}{\left(\sqrt[3]{abc}\right)^2} + \frac{bc}{\left(\sqrt[3]{abc}\right)^2} + \frac{ca}{\left(\sqrt[3]{abc}\right)^2} \iff$$

$$a^2 + b^2 + c^2 - 3\left(\sqrt[3]{abc}\right)^2 \geq \sqrt[3]{abc}\,(a + b + c) + ab + bc + ca \iff$$

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \sqrt[3]{abc}\left(a + b + c - 3\sqrt[3]{abc}\right).$$

$(SQM) \implies (T):$
Denote $a := x^2, b := y^2, c := z^2, d := t^2$ and due to homogeneity and symmetry of (T) suppose that $abcd = 1$ and $d = \min\{a, b, c, d\}$ .Then (T) becomes

$$a^2 + b^2 + c^2 + d^2 + 2 \geq ab + bc + cd + da + ac + bd \iff$$

$$a^2 + b^2 + c^2 - ab - bc - ca \geq d\,(a + b + c) - 2 - d^2. \qquad (T1)$$

Since $d = \min\{a, b, c, d\}$ and $abcd = 1$ then $abc = \dfrac{1}{d}$ and $1 \geq d^4 \implies d \leq 1.$
Due to (SQM)

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \sqrt[3]{abc}\left(a + b + c - 3\sqrt[3]{abc}\right) = \frac{a + b + c}{\sqrt[3]{d}} - \frac{3}{\sqrt[3]{d^2}}$$

and

$$a + b + c \geq 3\sqrt[3]{abc} = \frac{3}{\sqrt[3]{d}} \iff \sqrt[3]{d}\,(a + b + c) \geq 3$$

we have

$$a^2+b^2+c^2-ab-bc-ca-\left(d\,(a+b+c)-2-d^2\right)\geq$$

$$\frac{a+b+c}{\sqrt[3]{d}}-\frac{3}{\sqrt[3]{d^2}}-\left(d\,(a+b+c)-2-d^2\right)=\sqrt[3]{d}\,(a+b+c)\left(1-d\sqrt[3]{d}\right)+$$

$$2\sqrt[3]{d^2}+d^2\sqrt[3]{d^2}-3\geq\ 3\left(1-d\sqrt[3]{d}\right)+2\sqrt[3]{d^2}+d^2\sqrt[3]{d^2}-3=\sqrt[3]{d^2}\left(d^2-3\sqrt[3]{d^2}+2\right)=$$

$$=\sqrt[3]{d^2}\left(\sqrt[3]{d^2}-1\right)^2\left(\sqrt[3]{d^2}+1\right)\geq 0.$$

$(T)\implies(DLT):$
By setting in (T) $t=1, a=x^2, b=y^2, c=z^2$ we obtain

$$a^2+b^2+c^2+1+2\sqrt{abc}\geq ab+bc+ca+a+b+c$$

and since $2\sqrt{abc}\leq 1+abc$ then

$$ab+bc+ca+a+b+c\leq a^2+b^2+c^2+abc+2\iff$$

$$a^2+b^2+c^2+2abc+3\geq(1+a)(1+b)(1+c)\,.$$

$(DLT)\implies(B):$
See in the proof of (DLT).

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